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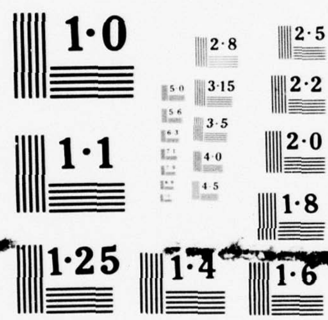


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# APPLICATION OF FUNCTIONAL ANALYSIS IN FLUID-MECHANICS.

BY


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AERODYNAMICS INSTITUTE UNIVERSITY OF NAPLES




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# APPLICATION OF FUNCTIONAL ANALYSIS IN FLUID-MECHANICS

prepared by

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## 0. FOREWORD

The work done during the period covered by the present FINAL Scientific reports pertains to the following topics related to the application of functional analysis in fluidmechanics:

1) Variational formulation of non-linear boundary value problems;

2) Closed splines.

The present Report is divided in two parts.

The first part describes more or less concisely, according to the needs, the work which has already been extensively reported in available publications. It consists of three chapters dealing with : a) A class of non linear boundary value problems; b) Applications of closed spline functions; c) Numerical sperimentations with closed splines.
The description of the nature of this work and of the publications it has originated is given in what follows.
As a natural development of the research program on applications of functional analysis to fluid-mechanics, the main objective of the next stage of the investigation was clearly to be identified with the extension of the results found for linear problems to as large a class of non-linear problems as possible.
A first positive step in this direction has been accomplished.

The following problem has been investigated and solved: variational formulation of the non-linear problem $A(u)=f(u)$ where A is a positive definite, formally self-adjoint operator in a suitably defined Hilbert space H of the functions ( $u$ ) and $f$ is a Frechlt differentiable non-linear operator defined in H.

A first version of a paper describing this work and co-authored by the principal investigator and by Dr. C. GOLIA has been presented at the Third Congress of the Italian Association for Theoretical and Applied Mechanics (AIMETA) held in Cagliari (Italy) during the month of October, 1976 [Ref. 1].

In the preprint of this paper acknowledgment to AFOSR support was inadvertedly omitted and only the sponsoring received by the second author was acknowledged. This error will be corrected in the second version of the paper (see later).

Further studies on the subject have led to a much improved formulation and solution of said problem. This has resulted in the paper:

1. L.G. NAPOLITANO, C. GOLIA: " Dual Variational Formulation of a non-Linear Boundary Value Problem"

which will be submitted for publication in an International Journal. Since the first version of the papar is available [Ref. 1] and the final version presents only formal improvements of the first one, only a concise description of the results needs to be given. This will be found in Part I of the present Report, chapter one.

The activity concerning closed splines has been concerned with:

i) working out applications of the closed spline functions introduced by the principal investigator [Ref. 2]

ii) carrying out numerical sperimentation with these closed splines

iii) defining and studying the properties of a new class of closed splines: the Hermite closed splines

The definition, characterization and properties of closed splines were described in the Final Scientific Report of the previous Grant AFOSR 76-2889 [Ref. 2]. These new classes of splines yielded interpolating functions for closed curves and possess all the properties of spline-functions. Conventional approaches still largely use conventional splines to interpolate closed curves: the analysis reported in [3] illustrates the many shortcomings of such an approach and shows how markedly

superior is the use of closed splines. To clearly substantiate these theoretical predictions it was felt appropriate to work out some examples of applications.
The problem of interpolating significant airfoil shapes was considered and results obtained from classical and closed-splines formulations were compared.
These examples of applications of closed-splines are described in Part I of the present report, chapter two. They were included in the paper:

2. L.G. NAPOLITANO, V. LOSITO " The closed splines functions"

which has been accepted for publication in the "Journal of Computational Physics".
Once the theory of closed spline functions had been developed, the next stage of analysis was addressed. It concerned a number of questions related to the optimal use of closed splines in a computational scheme and called for extensive numerical sperimentation.
This phase of the work is completed and its results are described in the following paper which has been accepted for presentation at the Congress of the Italian Association for Aeronautics and Astronautics (AIDAA) which will be held in Milan, during the last week of Septebmer 1977:

3. L.G. NAPOLITANO, V. LOSITO and A. VITIELLO "Numerical sperimentation with Closed Splines "

For this reason only a concise summary of theproblems addressed and of the results obtained will be given in Part I of this report, Chapter three.

The second part of this report describes in extenso the third development of the research work on closed spline, i.e. the one related to Hermite closed splines.

# PART I SUMMARY OF WORK ALREADY PUBLISHED

## 1.1. INTRODUCTION

As mentioned in the Foreword, this first part of the report describes concisely work which is already available or will shortly become available in published form.

## 1.2. DUAL VARIATIONAL FORMULATION OF A CLASS OF NON LINEAR BOUNDARY VALUE PROBLEMS.

The following class of non-linear boundary value problems $A(u) = f(u)$ has been considered, where A is a positive definite, formally self-adjoint operator in a suitably defined Hilbert space H of the functions ( u ) and f is a Frechet differentiable non-linear operator defined on H.
First the conditions on f which allow a strong variational formulation of the problem have been determined. It has been found that these conditions reduce to the symmetry of the non-linear mapping f (u).

Subsequently the hybrid strong variational formulation has been derived from which the dual variational formulations have been obtained as particular cases [see Refs (5), (6) originated under a previous grant for the meaning which is ascribed to hybrid functionals in the subject context].
Finally, upper and lower bounds for the exact solution and error estimates for the approximate solutions have been found.

The approach used is based on the search of an hybrid functional by means of integration, in a cartesian product of two suitable Hilbert spaces, of a gradient. Imposition of the integrability conditions and of appropriate stationarily conditions leads to the requirements on A and f that must be satisfied for a strong variational formulation to be possible. The subsequent developments are based on more or less straight-forward application of functional analysis results which are either classical or were obtained during previous grants: [Refs. (5) ,(6)]

1.3. EXAMPLES OF APPLICATION OF CLOSED SPLINES.

A number of numerical techniques for the solution of planar inviscid flow fields around airfoils require knowledge of the parametric equations $x(\tau), z(\tau)$ of the airfoil's profile in terms of its arc length $\tau$. These equations must be found by interpolating the data $z_\pm(x_i)$ representing the ordinates of the upper and lower profile at a number of stations $x_i$.

The arc length $\tau$ is, of course, not known a priori and the problem has to be solved iteratively. A possible procedure is as follows.

Assume as curve parameter the length $w$ measured along the polygonal connecting the assigned profile points $P_i$ to obtain, from the data $z_i = z(x_i)$ two sets of values:

$$\{w_i, x_i\} ; \quad \{w_i, z_i = z(x_i)\}$$

where $w_i = w(P_i)$

From these two sets construct the two interpolating functions $x(w)$; $z(w)$ and compute

$$w^{(1)}(w) = \int_0^w \left[ \left(\frac{dz}{dw}\right)^2 + \left(\frac{dx}{dw}\right)^2 \right]^{1/2} dw$$

The quantity $w^{(1)}$ [which would give the profile's arc length if $x(w)$, $z(w)$ where the exact parametric equations of the profile in terms of $w$] is taken as the new curve parameter and the cycle is repeated. From the new sets of values $\{W_i^{(1)}, x_i\}$; $\{W_i^{(1)}, z_i = Z(x_i)\}$ [ where $W_i^{(1)} = W^{(1)}(W_i)$] two new interpolating functions $x = x[W^{(1)}]$; $z = z[W^{(1)}]$ are constructed, a new curve parameter $W^{(2)}(W^{(1)})$ determined and so on.

The iteration terminates when $|W^{(r)} - W^{(r-1)}|_{max}$ is smaller than a pre-assigned small quantity for then the r-th iterate $w^{(r)}$ pratically coincides with the arc length ( $\tau$ ) and the functions $x[W^{(r)}]$, $z[W^{(r)}]$ likewise coincide with the interpolating functions $x(\tau)$, $z(\tau)$ .

At each cycle the procedure requires the construction of two interpolating functions $x[W^{(j)}]$, $z[W^{(j)}]$. In the examples to be discussed here conventional and closed cubic splines (q=2) with several values for n are considered.

The interpolating functions based on conventional splines are denoted without any subscript and those based on closed splines with a subscript (c). To simplify the notations, primes will always denote derivatives with respect to the pertinent parameter.

According to the results of the theory, [see Refs (2) ,(3)] for any (n) and at each cycle: $x_c'$ , $x_c''$, $z_c'$, $z_c''$ will be continuous everywhere, x' and z' will be discontinuous at the closure point, x'' and z'' will be continuous everywhere but will vanish at the closure point. Hence, at this point, the conventional splines will yield an infinite radius of curvature R.

To facilitate comparison with exact values a family of elliptic profiles is considered, which is defined by the equations $x = \frac{1}{2}(1-\cos\theta)$, $z = \rho/2 \sin\theta$ where $\theta$ is the polar angle and $0.5 \leq \rho \leq 1$ is the thickness ratio. The closure point is the point corresponding to $\theta = 0$ and the other points $P_i$ correspond to constant increments in $\theta$.

The question of the optimal location of the points $P_i$ is an interesting one and becomes very important for low values of $\rho$. It will be addressed in the next paragraph.

The discussion will be limited to the final interpolating parametric curves x ($\tau$), z ($\tau$) as in all cases considered the iteration converged very rapidly (at most two cycles were needed).

Due to the symmetries of the airfoils and to the choice of the points $P_i$, the discontinuity in the drivative z' at the closure point turns out to be zero whereas that of the derivative x' is equal to twice the value of x' at $\theta = 0^+$.

The values of the discontinuities x' at the closure point for different values of (n) and for elliptic airfoils ($\rho$ = 1, .75, .5) are shown in Tab. 1. Obviously they do decrease as n increases, but the point here is that a much better result (vanishing discontinuities) can be obtained at no cost (i.e. without having to increase the number of subintervals) with the use of closed spline functions. Tab. I also shows that things get worse as $\rho$ decreases: the same trend will be found in all other features.

Values of the derivatives $dx/d\tau$ and $dz/d\tau$ are shown in Tab. IIa through IIc for the same three values of $\rho$ and n=8. Due to the symmetry of the profiles only values for $0 \leq \theta \leq \pi$ are listed: the first column gives the exact values, the other two columns give the values obtained with the closed spline and the classical spline, respectively.

As anticipated in Refs (2), (3) :

i) the discontinuity at the closure point is associated with poor accuracy in its neighborhood. Thus, since $\delta x' \neq 0$ whereas $\delta z' = 0$ at the closure point, the accuracy in x' is worse than in z' in its neighborhood. Tables II show that both accuracies decrease as $\rho$ decreases and that the results yielded by the closed splines are far superior to those given by the conventional ones;

ii) the discontinuity at the closure point influences the behaviour of the interpolating function throughout. Thus, for instance, the interpolating functions based on classical spline fail to exhibit the symmetry properties $dx/d\tau(\pi/_2 + \theta) = dx/d\tau(\pi/_2 - \theta)$ and $dz/d\tau(\pi/_2 + \theta) = - dz/d\tau(\pi/_2 - \theta)$, which are instead satisfied by the interpolating functions based on closed spline;

iii) as one moves away from the closure point the "smoothness properties" of the interpolating functions based on classical spline tend to approach, at a rate depending on the curve to be interpolated, those of the interpolating functions based on closed splines.

Exact and approximate values of the radius of curvature R were also analyzed for $\rho = 1$ and $\rho = 0.5$ with $n=8$. The analysis confirmed that:

1) the classical spline interpolating function is vastly off in the neighborhood of the closure point, whereas the closed spline gives there very satisfactory values;

2) the performance of the classical spline approaches that of the closed spline sufficiently far away from the closure point.

TAB. I

Discontinuities $\delta x'$ at the closure point for different values of n when elliptic airfoils are approximated by means of classical splines.

| Elliptic Airfoils Percentual Thickness | $\delta x'$ | | |
|---|---|---|---|
| $\rho$ | n=8 | n=16 | n=32 |
| 1 | .4944 | .2322 | .1139 |
| .75 | .7480 | .3306 | .1546 |
| .50 | 1.1310 | .5546 | .2444 |

TAB.IIa

Comparison among the values of the derivatives $dx/d\tau$ and $dz/d\tau$ - $n = 8$ - Circular Airfoil ($\rho = 1$)

| | $dx/d\tau$ | | | $dz/d\tau$ | | |
|---|---|---|---|---|---|---|
| $\theta$ | EXACT VALUE | CLOSED SPLINE | CLASSICAL SPLINE | EXACT VALUE | CLOSED SPLINE | CLASSICAL SPLINE |
| 0 | 0 | 0.0000 | 0.2472 | 1 | 0.9983 | 1.0140 |
| $\pi/8$ | 0.3827 | 0.3832 | 0.3457 | 0.9239 | 0.9252 | 0.9388 |
| $\pi/4$ | 0.7071 | 0.7059 | 0.6411 | 0.7071 | 0.7059 | 0.7129 |
| $3\pi/8$ | 0.9239 | 0.9252 | 0.[illegible]345 | 0.3827 | 0.3832 | 0.3809 |
| $\pi/2$ | 1 | 0.9983 | 1.0140 | 0 | 0.0000 | 0.0026 |
| $5\pi/8$ | 0.9239 | 0.9252 | 0.9218 | -0.3827 | -0.3832 | -0.3825 |
| $3\pi/4$ | 0.7071 | 0.7059 | 0.7018 | -0.7071 | -0.7059 | -0.7051 |
| $7\pi/8$ | 0.3827 | 0.3832 | 0.3844 | -0.9239 | -0.9252 | -0.9256 |
| $\pi$ | 0 | 0.0000 | 0.0000 | -1 | -0.9983 | -0.9990 |

TAB.IIb

Continued: Elliptic Airfoil - ( $\rho = 0.75$ )

| $\theta$ | $dx/d\tau$ EXACT VALUE | $dx/d\tau$ CLOSED SPLINE | $dx/d\tau$ CLASSICAL SPLINE | $dz/d\tau$ EXACT VALUE | $dz/d\tau$ CLOSED SPLINE | $dz/d\tau$ CLASSICAL SPLINE |
|---|---|---|---|---|---|---|
| 0 | 0 | 0.0000 | 0.3740 | 1 | 0.9560 | 0.9780 |
| $\pi/8$ | 0.4834 | 0.4900 | 0.4528 | 0.8754 | 0.8710 | 0.8952 |
| $\pi/4$ | 0.8000 | 0.7920 | 0.6893 | 0.6 | 0.6320 | 0.6440 |
| $3\pi/8$ | 0.9550 | 0.9560 | 0.9702 | 0.2967 | 0.2790 | 0.2751 |
| $\pi/2$ | 1 | 1.0089 | 1.0340 | 0 | 0.0000 | -0.0040 |
| $5\pi/8$ | 0.9550 | 0.9560 | 0.9493 | -0.2967 | -0.2790 | -0.2775 |
| $3\pi/4$ | 0.8 | 0.7920 | 0.7860 | -0.6 | -0.6320 | -0.6310 |
| $7\pi/8$ | 0.4834 | 0.4900 | 0.5061 | -0.8754 | -0.8710 | -0.8754 |
| $\pi$ | 0 | 0.0000 | 0.0000 | -1 | -0.9560 | -0.9560 |

TAB.IIc

Continued : Elliptic Airfoil - ( $\rho = 0.5$ )

| | $dx/d\tau$ | | | $dz/d\tau$ | | |
|---|---|---|---|---|---|---|
| $\theta$ | EXACT VALUE | CLOSED SPLINE | CLASSICAL SPLINE | EXACT VALUE | CLOSED SPLINE | CLASSICAL SPLINE |
| 0 | 0 | 0.0000 | 0.5655 | 1 | 0.8424 | 0.8710 |
| $\pi/8$ | 0.6380 | 0.6858 | 0.6170 | 0.7701 | 0.7656 | 0.7918 |
| $\pi/4$ | 0.8944 | 0.9351 | 0.7710 | 0.4472 | 0.5353 | 0.5539 |
| $3\pi/8$ | 0.9792 | 0.9708 | 0.9973 | 0.2028 | 0.1665 | 0.1623 |
| $\pi/2$ | 1 | 0.9826 | 1.0230 | 0 | 0.0000 | -0.0053 |
| $5\pi/8$ | 0.9792 | 0.9708 | 0.9621 | -0.[illegible]028 | -0.1665 | -0.1653 |
| $3\pi/4$ | 0.8944 | 0.9351 | 0.9272 | -0.4472 | -0.5353 | -0.5343 |
| $7\pi/8$ | 0.6380 | 0.6858 | 0.6882 | -0.7701 | -0.7656 | -0.7661 |
| $\pi$ | 0 | 0.0000 | 0.0000 | -1 | -0.8424 | -0.8434 |

## 1.4. NUMERICAL SPERIMENTATION WITH CLOSED SPLINES.

As discussed in the previous paragraph the performance of closed-splines interpolating functions tended to become worse and worse as the percentual thickness ($\varrho$) of the elliptical airfoils decreased.
This tendency is to be ascribed to the fact that, as $\varrho$ decreases, the curvatures at the leading (and trailing) edges of the airfoil increase whereas that of the central part decreases. Hence there was a clear indication that an uniform polar angle spacing of the data points $P_i$ is not always adequate.

The optimal distribution of a prescribed total numer (n) of points $P_i$ constituted the first objective of numerical sperimentation with closed spline.

The obvious speculation was that the points $P_i$ had to be "concentrated" in the large curvature region. Numerical sperimentation confirmed this speculation and showed a marked improvement in the accuracy of the interpolating functions with the use of cubic closed splines.
These encouraging results point to a very promising area of further theoretical research, namely that of optimal location of data points. Some preliminary suggestions on possible theoretical approaches are given in the paper mentioned in the Foreword. [Ref (4)].
The relocation remedy is naturally applicable only when one can prescribe arbitrarily the position of the points $P_i$.
When this is not the case, i.e. when both the total number of points and their locations are given, the alternative may seem to be an increase in the degree ($q$) of the closed spline.

Extensive numerical sperimentation proved however that, in general, this is not so. It turned out, indeed, that for each number ( n ) of points $P_i$ (and given distribution of them) there is an optimal value $q_o$ of the degree of the spline that gives the "best" approximation. Or, equivalently, for any given degree of the spline there is, for a given distribution, and optimal value $n_o$ of point $P_i$ which yields the "best" approximation.

The optimal value $n_o(q)$ increases with q. Thus, as stated, it is a fallacious illusion the one that, for a given fixed number of points $P_i$ and a given distribution of them, hopes to increase the accuracy of the interpolation by increasing the degree of the spline. This will be the case if and only if the initial degree of the spline is less than the optimal one corresponding to the given number (n) of points.

This being the state of affairs the true problem is then how to improve on the accuracy of the approximation when both the number of points (nad their distribution) and the degree of the spline (at its optimal value) are fixed.

Also for this problem some preliminary suggestions are offered in the paper mentioned in the Foreword [Ref.(4)]

## 1.5. REFERENCES

1. L.G. NAPOLITANO, C. GOLIA: "Dual Variational Formulation of non linear Problems in Fluid Dynamics" III Congress of the Italian Association for theoretical and appllied Mechanics (AIMETA), Cagliari, October 1976.

2. Final Scientific Report Grant AFOSR 76-2889; I.A. Report No. 235, July 1976

3. L.G. NAPOLITANO, V. LOSITO: "The Closed Spline Functions" to appear in "Journal of Computational Physics"

4. L.G. NAPOLITANO, V. LOSITO, A. VITIELLO: "Numerical sperimentation with closed splines" to be presented at the Congress of the Italian Association for Aeronautics and Astronautics, Milan, September 1977

5. L.G. NAPOLITANO: "On the functional Analysis Derivation and Generalization of Hybrid Methods", Meccanica, Vol. 10, No. 3 Sept. 1975.

6. L.G. NAPOLITANO: "Functional Analysis Approach for the Derivation of Hybrid Functionals" in Application of Methods of Functional Analysis to Problems in Mechanics, Lecture Notes in Mathematics, 503, Springer Verlag, 1976

# PART II

## HERMITE CLOSED SPLINE FUNCTIONS

by

L. G. NAPOLITANO

### 1. INTRODUCTION

In computational aerodynamics it is often necessary to solve interpolating problems related to "airfoils", i.e. to closed curves.
Using classical spline functions to interpolate airfoil's ordinates prescribed on a finite set of point is unsatisfactory, and on many important accounts [1] and a markedly better approach is afforded by the closed spline functions [1].

A similar situation arises when the data to be interpolated represent values of the functions and of its first derivative at a given set of points.

Consider indeed the problem solved by classical Hermite spline functions.

Given a closed interval, reduced through suitable normalization to $[0,1]$, (n+1) points $P_i(\tau_i)$ with:

$$0 = \tau_1 \leq \tau_2 < \text{------} \tau_n < \tau_{n+1} = 1$$

and (2n) real numbers (ri); (ri') ( $1 \leq i \leq n+1$) the interpolating Hermite spline function of order $q < n+1$ corresponding to the (n + 1) points $P_i$ and to the sets $\{r_i\}, \{r_i'\}$ is the unique function $\tilde{\sigma} \in H^q[0,1]$ which solves the following minimization problem:

$$\begin{cases} \min\limits_{f \in H^q} \int_0^1 [f^{(q)}(\tau)]^2 d\tau \\ \\ f(\tau_i) = r_i \\ \qquad\qquad\qquad (1 \leq i \leq n+1) \\ f'(\tau_i) = r_i' \end{cases} \tag{1.1}$$

where $f'(\tau)$ denotes the first derivative of $f(\tau)$ and $H^q[0,1]$ is the Hilbert space of real functions $f(\tau)$ defined on $[0,1]$ and having a square-integrable q-th derivative $f^{(q)}(\tau)$.

The space $H^q[0,1]$ quantifies the notion of "degree of smoothness" of the interpolating curve.

The Hermite interpolating splines belong to the $2(n+1)$ - dimensional subspace $\tilde{S} \subset H^q[0,1]$ of real functions $\tilde{s}(\tau)$ defined on $[0,1]$ and such that [2] :

a) $\tilde{s}(\tau)$ is a polinomial of degree (2q-1) in each of the open intervals $]\tau_i, \tau_{i+1}[$ ; $(1 \leq i \leq n)$

b) $\tilde{s}(\tau)$ and its first (2q-3) derivatives are continuous on $[0,1]$

c) the derivatives of $\tilde{s}(\tau)$ from orders $q$ to $(2q-1)$ vanish at the points $\tau_1 = 0$, $\tau_{n+1} = 1$.

Such an Hermite spline is given by [2]:

$$\tilde{s}(\tau) = \sum_{j=0}^{q-1} \tilde{\beta}_j \frac{\tau^j}{j!} + \sum_{i=1}^{n+1} \left[ \frac{\tilde{\lambda}_i (\tau - \tau_i)_+^{2q-1}}{(2q-1)!} + \frac{\tilde{\lambda}_i' (\tau - \tau_i)_+^{2q-2}}{(2q-2)!} \right] \tag{1.2}$$

where:

$$(\tau - \tau_i)_+^k = \begin{cases} (\tau - \tau_i)_+^k & \text{if } (\tau - \tau_i) \geq 0 \\ 0 & \text{if } (\tau - \tau_i) < 0 \end{cases} \tag{1.3}$$

and the $2(n+1)$ coefficients $\tilde{\lambda}_i$, $\tilde{\lambda}_i'$ satisfy the q equations:

$$\sum_{i=1}^{n+1} \left[ \tilde{\lambda}_i (\tau_i)^k + k \tilde{\lambda}_i' (\tau_i)^{k-1} \right] = 0 \qquad [0 \leq k \leq q-1] \tag{1.4}$$

(with the convention that $k\, \tau_i^{k-1} = 0$ for $k = 0$).

By imposing the $2(n+1)$ additional requirements

$$\tilde{s}(\tau_i) = z_i \; ; \; \tilde{s}'(\tau_i) = z_i' \; ; \; 1 \leq i \leq n+1 \tag{1.5}$$

the $[q + 2(n+1)]$ coefficients $\tilde{\beta}_j$, $\tilde{\lambda}_i$, $\tilde{\lambda}_i'$ are uniquely determined [2] and so it is the resulting Hermite interpolating function $\tilde{\sigma}(\tau)$

solving problem (1.1)

If the data to be interpolated correspond to a closed curve then: $P_1 = P_{n+1}$ (closure point); $\tau_1 = \tau_{n+1}$, $\tau_1' = \tau_{n+1}'$ and $\tau$ is to be interpreted as the curvilinear coordinate of a sufficiently smooth and regular closed contour C normalized with respect to its length [1].
If, to interpolate these data, one uses the classical Hermite function $\tilde{\sigma}(\tau) = \tilde{s}(\tau)$ [ where $\tilde{s}(\tau)$ is defined by $(1.2)$ through (1.5) with $\tau_1 = \tau_{n+1}$; $\tau_1' = \tau_{n+1}'$ then $\tilde{\sigma}(\tau)$ will indeed have the above mentioned spline properties in each of the (n) open sub-intervals $]\tau_i, \tau_{i+1}[$, $(1 \le i \le n+1)$
However, at the closure point $P_1 = P_{n+1}$:

i) the conditions $\tau_1 = \tau_{n+1}$; $\tau_1' = \tau_{n+1}'$ guarantee only the continuity of the function and of its first derivative; the other derivatives, up to the order (q-1), will not be continuous [unless the location of points $P_i$ and the data sets $\{\tau_i\}$, $\{\tau_i'\}$ satisfy particular symmetry conditions ], their discontinuities being uniquely determined; [ upon the uniqueness of the solution of problem (1.1)];

ii) the subsequent derivatives, up to the order (2q-3), are continuous but vanish identically [upon the propertis (b) and (c) of $\tilde{s}(\tau)$];

iii) the discontinuity of the remaining derivatives are likewise uniquely determined: that of the (2q-2)-th derivative being equal to $(\tilde{\lambda}_1' - \tilde{\lambda}_{n+1}')$ and that of the (2q-1)-th derivative being equal to $(\tilde{\lambda}_1 - \tilde{\lambda}_{n+1})$.

Thus, as for the problem considered in [1], the use of conventional Hermite spline functions for closed curves presents a number of shortcomings. Unwanted sicontinuites are introduced at the closure point and the vanishing of the derivatives from orders q to (2q - 3) may be too penalizing, especially for low values of q. In addition, always because of point i)

above, the [illegible]terpolating function does not belong to $H^q$ (C). Hence, its "degree of smoothness" is not characterized and, perhaps more critically, $\tilde{\sigma}$ does not satisfy any minimum problem.
These shortcomings are eliminated if one uses the notion of closed splines introduced by the author in [1] and constructs a new class of such splines, the Hermite closed splines, constituting, as the others, a subspace of $H^q$ (C). The present paper is devoted to this task.

The approach to be used to construct and study Hermite closed splines is based on the abstract-space spline theory detailed in [2] and it is perhaps appropriate tojustify its use.
One might argue that all one needs is the introduction of (q-2) additional parameters to impose the missing continuity of the derivatives from orders 2 to (q - 1). This speculation is substantially correct. The resulting interpolating spline would have a total of $[q+2(n+1)+q-2]-2(q+n)$ parameters and would essentially coincide with what we call a Hermite closed interpolating spline. However one would feel at loss as to how actually to proceed, and, more fundamentally, questions related to existence and uniqueness of such interpolating functions would be left unanswered. Furthermore, it would be difficult to ascertain and eventually establish the equivalence between the interpolation problem and a minimization problem, to identify the subspace generated by these "splines", to describe their extremal properties and so on.
All these questions, by contrast, are most simply resolved by using abstract space spline theory.
As in [1] , to facilitate a more wide-spread comprehension, the definition characterization and properties of Hermite closed-splie functions are first stated without proof in paragraph (2).
Existence uniqueness characterization extremal and other relevant properties are proven in paragraph (3), wherein for completeness' sake, the pertinent needed results from the Hilbert space theory of spline-functions

are also concisely recalled.

As this paper employs, with few variants, the same approach developed in [1] it will be rather more concise and reference is to be made to [1] for missing details.

## 2. DEFINITIONS, CHARACTERIZATION AND PROPERTIES OF HERMITE CLOSED SPLINES

### a) Definitions

Let C be a sufficiently smooth closed contour. Denote by $\tau$ the curvilinear coordinate along C measured from an arbitrary initial point $P_1$ and normalized with respect to the length of C, so that $0 \leq \tau \leq 1$. $P_1$, to be referred to as the closure point on the contour, is characterized by either value $\tau = 0^+$; $\tau = 1^-$.

Consider (n) arbitrary successive points $P_i \, (1 \leq i \leq n)$ on C: let $\tau_i$ be their curvilinear coordinates with:

$$0 = \tau_1 < \tau_2 < \tau_3 \cdots\cdots < \tau_n < 1$$

and prescribe (2n) real numbers $r_i$, $r_i'$.

<u>The Hermite closed interpolating spline</u> $\sigma$ of degree (q) corresponding to the (n) triplets $(\tau_i, r_i, r_i')$ $(1 \leq i \leq n)$ is <u>defined</u> as the unique element of $H^q$ (C) such that:

$$\oint_C [\sigma^{(q)}]^2 d\tau = \min_{f \in I} \oint_C [f^{(q)}(\tau)]^2 d\tau \qquad (2.1)$$

$$I = \left\{ f \in H^q(C) \mid f(\tau_i) = r_i;\ f'(\tau_i) = r_i';\ 1 \leq i \leq n \right\}$$

In order for $\sigma \in H^q(C)$ to be such a Hermite closed interpolating spline it is necessary and sufficient that:

a) $\sigma$ be a polynomial of degree (2q-1) in each open sub-interval $]0=\tau_1, \tau_2[, ---- ]\tau_i, \tau_{i+1}[, ---- ]\tau_n, 1[;$

b) $\sigma$ be continuous, on C, together with its first (2q - 3) derivatives, i.e.

$$\left.\begin{array}{l} \sigma^{(k)}(0^+) = \sigma^{(k)}(1^-) \\ \sigma^{(k)}(\tau_i^+) = \sigma^{(k)}(\tau_i^-) \quad ; \forall i \in [2, n] \end{array}\right\} \forall k \in [0, 2q-3]$$

c) $\sigma$ be such taht

$$\sigma(\tau_i) = z_i \quad ; \quad \sigma'(\tau_i) = z_i' \quad ; \quad \forall i \in [1, n]$$

The set of functions $s(\tau) \in H^q(C)$ satisfying the first two conditions constitutes a subspace $S_{HC}$ [the space of Hermite closed spline functions corresponding to the set $\{\tau_i\}$] of dimension (2n).

b) Characterization

An element of $H^q(C)$ belongs to the subspace $S_{HC}$ of Hermite closed splines corresponding to $\{\tau_i\}$ if it is representable as

$$s(\tau) = \sum_{j=0}^{2q-1} \beta_j \frac{\tau^j}{j!} + \sum_{i=1}^{n} \left[ \frac{\lambda_i (\tau - \tau_i)_+^{2q-1}}{(2q-1)!} + \frac{\lambda_i' (\tau - \tau_i)_+^{2q-2}}{(2q-2)!} \right] \quad (2.2)$$

where the (2n) coefficients $\lambda_i, \lambda_i'$ and the (2q) coefficients $\beta_j$ satisfy the following (2q) equations:

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad (2.3)$$

$$\sum_{j=0}^{t-1} \left\{ \frac{\beta_{2q-t+j}}{(j+1)!} + (-1)^{t+j} \sum_{i=1}^{n} \left[ \frac{\lambda_i \tau_i^{t-j}}{(t-j)!} - \frac{\lambda_i' \tau_i^{t-j-1}}{(t-j-1)!} \right] \right\} = 0 \qquad (2.4)$$

The functions $(\tau-\tau_i)_+^k$ is defined by equation (1.3): it is continuous on C together with its first (k - 1) derivatives and its k-th derivative is discontinuous at $\tau=\tau_i$, the discontinuity being equal to $(k!)$ [2].

The (2q-1)-th and (2q-2)-th derivatives of s ( ) are given by:

$$s^{(2q-1)}(\tau) = \beta_{2q-1} + \sum_{i=1}^{n} \lambda_i (\tau-\tau_i)_+^0$$

$$s^{(2q-2)}(\tau) = \beta_{2q-2} + \beta_{2q-1}\tau + \sum_{i=1}^{n} \left[ \lambda_i (\tau-\tau_i)_+ + \lambda_i' (\tau-\tau_i)_+^0 \right]$$

Hence:

i) the (2q-1)-th derivative of s ( $\tau$ ) is piecewise constant in each open sub-interval. Specifically it is equal to ( $\beta_{2q-1} + \sum_{j=1}^{i} \lambda_j$ ) in $]\tau_i, \tau_{i+1}[, (1 \le i \le n-1)$ and to $\beta_{2q-1}$ in $]\tau_n, 1[$. Thus the (n) coefficients $\lambda_i$ represent the values of the discontinuity:

$$\delta s^{(2q-1)}(P_i) = s^{(2q-1)}(\tau_i^+) - s^{(2q-1)}(\tau_i^-)$$

at the Points $P_i$. Notice that for $i=1$ it is $\tau_1^+ = 0^+;\ \tau_1^- = 1^-$ ; thus

$$\delta s^{(2q-1)}(P_1) = s^{(2q-1)}(0^+) - s^{(2q-1)}(1^-) = \beta_{2q-1} + \lambda_1 - \beta_{2q-1} = \lambda_1$$

ii) The (2q-2)-th derivative of s ($\tau$) is piecewise linear in each open sub-interval. Specifically, it is equal to

$$\beta_{2q-2} + \beta_{2q-1}\tau + \sum_{j=1}^{i} \left[ \lambda_i (\tau-\tau_i) + \lambda_i' \right]$$

in $]\tau_i, \tau_{i+1}[, (1 \le i \le n-1)$ and, on account of eqs (2.3) and (2.4) for $r = 1$, to

$$\beta_{2q-2} + \beta_{2q-1} (\tau - 1)$$

in $]\tau_n, 1[$ . The discontinuities of $s^{(2q-2)}$ at the points $P_i$ are thus given by:

$$\delta s^{(2q-2)}(P_i) = \lambda_i' \qquad (2 \leq i \leq n)$$
$$\delta s^{(2q-2)}(P_1) = \beta_{2q-2} + \lambda_1' - \beta_{2q-2} = \lambda_1'$$

Hence the coefficients $\lambda_i'$ represent the discontinuities of the (2q-2)-th derivatives of s ($\tau$) at the points $P_i$ $[1 \leq i \leq n]$.

Notice how the closure point ( $P_1$ ) is in no way differentiated from the other (interior) points.

Given (2n) arbitrary real numbers $t_i, t_i'$, thre is a unique Hermite closed spline corresponding to the set $\{\tau_i\}$ such that:

$$s(\tau_i) = t_i \; ; \quad s'(\tau_i) = t_i' \; ; \quad (1 \leq i \leq n) \tag{2.5}$$

Consequently, the system of 2(n + q) equations (2.3), (2.4), (2.5), linear in $\beta_q, \lambda_i, \lambda_i'$ admits of a unique solution and serves to determine the 2 (n + q) coefficients appearing in the unique interpolating Hermite spline function: $\sigma(\tau)$ which solves the minimum problem (2.1).

c) <u>Properties</u>

For any element $s \in S_{HC}$ and any function $f \in H^q(C)$ it is:

$$\oint_C s^{(q)}(\tau) f^{(q)}(\tau)\, d\tau = (-1)^q \sum_{i=1}^{n} \left[ \lambda_i f(\tau_i) - \lambda_i' f'(\tau_i) \right] \tag{2.6}$$

In particular:

$$\oint_C s^{(q)}(\tau) f^{(q)}(\tau)\, d\tau = 0 \tag{2.7}$$

$$\forall f \in I_0 = \{ f \in H^q(C) \mid f(\tau_i) = f'(\tau_i) = 0\ ;\ \forall i \in [1,n] \}$$

The minimization problem (2.1) solved by $\sigma(\tau)$ is a particular case of the following more general extremal properties.

Given (n) arbitrary but fixed $\tau_i$'s, if $\sigma$ is the unique element of $S_{HC}$ such that $\sigma(\tau_i) = z_i\ ;\ \sigma'(\tau_i) = z_i'$ then,

A) For any $f \in I$:

$$\oint_C [\sigma^{(q)} - f^{(q)}]^2\, d\tau = \min_{s \in S_{HC}} \oint_C [s^{(q)} - f^{(q)}]^2\, d\tau \tag{2.8}$$

and any other element $\bar{\sigma}$ of $S_{HC}$ having this property differs from $\sigma$ by a constant;

B) for any $s \in S_{HC}$:

$$\oint_C [\sigma^{(q)} - s^{(q)}]^2\, d\tau = \min_{f \in I} \oint_C [f^{(q)} - s^{(q)}]^2\, d\tau \tag{2.9}$$

and $\sigma$ is the unique element of $I$ having this property.

If the r.h.s. of equations (2.4) is denoted by $L_z$, the discontinuity $\delta s^{(k)}(P_1)$ of the k-th derivative of $s$ at the closure point P, is given by

$$\delta s^{(k)}(P_1) = -L_{2q-k-1}$$

Hence of the (2q-1) homogeneous conditions given by equations (2.4) only the last (2q-2) correspond to the required continuity conditions at the closure point. The remaining one (for = 1), which reads:

$$\beta_{2q-1} - \sum_{i=1}^{t} \lambda_i \tau_i + \sum_{i=1}^{t} \lambda_i' = 0 \tag{2.10}$$

simply expresses, as equation (2.3), obvious congruence conditions for the discontinuities.

Indeed, if $h(\tau)$ is any function discontinuous at the point $P_i$ and if we let:

$$\Delta_i h = h(\tau_{i+1}^-) - h(\tau_i^+)$$

$$\delta h(P_{i+1}) = h(\tau_{i+1}^+) - h(\tau_{i+1}^-)$$

then,

$$h(\tau_{i+1}^+) - h(\tau_i^+) = \Delta_i h + \delta h(P_{i+1})$$

so that, by summing for (i) from ( 1 ) to (n) and considering that, by the definition of the closing point, $P_{n+1} = P_1$; $\tau_{n+1}^+ = \tau_1^+ = 0^+$ one obtains that the discontinuities $\delta h(P_i)$ must satisfy the "congruence" condition:

$$\sum_{i=1}^{n} \Delta_{i-1} h + \delta h(P_i) = 0$$

which is really checked to reduce to equation (2.3) for $h(\tau) = s^{(2q-1)}(\tau)$ ~~and~~ $[\delta h(P_i) = \lambda_i]$ and to equation (2.10) for $h(\tau) = s^{(2q-2)}(\tau)$, $[\delta h = \lambda_i']$ once it is accounted for the fact that:

$$\Delta_i\, s^{(2q-2)}(\tau) = s_i^{(2q-1)}(\tau_{i+1} - \tau_i) =$$

$$= \left(\beta_{2q-1} + \sum_{j=1}^{i} \lambda_j\right)(\tau_{i+1} - \tau_i)$$

where $s_i^{(2q-1)}$ is the constant value of $s^{(2q-1)}$ in the subinterval $]\tau_i, \tau_{i+1}[$ and $\tau_{n+1} = 1$; $\tau_1 = 0$.

## 3. HERMITE CLOSED SPLINES: PROOFS OF EXISTENCE UNIQUENESS AND CHARACTERIZATION

Proofs of the statements made in the previous paragraphs hinge on the Hilbert space formulation of spline-functions theory detailed in [2].

For the sake both of self-completeness and of ready reference the basic results, as directly specialized to the present needs, are briefly summarized. A slightly more general formulation was given in [1] : greater details are to be found in [2].

If C is a sufficiently smooth and regular closed contour, consider the two Hilbert spaces $X = H^q(C)$ and $Y = H^0(C)$ with their standard inner products [denoted by $<\ ,\ >_{H^q}$ and $<\ ,\ >_{H^0}$] and the linear continuous operator $D_q : H^q(C) \to H^0(C)$ where $D_q$ denotes the q-th derivative with respect to the previously defined curvilinear coordinate $\tau \in [0, 1]$, normalized with respect to the length of C.

Finally let (2n) functionals $K_i ; K_i' : H^q(C) \to R$ be defined by:

$$
\begin{aligned}
< K_i , x >_{H^q} &= x(\tau_i) = t_i \qquad && x \in X = H^q(C) \\
< K_i' , x >_{H^q} &= x'(\tau_i) = t_i' \qquad && 1 \le i \le n
\end{aligned}
\tag{3.1}
$$

with:

$$0 = \tau_1 < \tau_2 < \text{------} < \tau_n < 1$$

By letting $Z \subset R^{2n}$, a finite dimensional subspace, and

$$z = (t_1, t_2, ---, t_n, t_1', t_2', ---, t_n') \in Z$$

equations (3.1) define an operator A: $H^q(C) \to Z$ and, for any fixed $z$, a subset $I_z$ of X, supposed non empty:

$$I_z = \{ x \in X \mid Ax = z \} \tag{3.2}$$

The null spaces of the operators A and $D_q$ are given by:

$$N(A) = I_0 = \{ x \in H^q(C) \mid \langle K_i, x \rangle_{H^q} = \langle K_i', x \rangle_{H^q} = 0 ;\ 1 \le i \le n \}$$

$$N(D_q) = \{ x \in H^q(C) \mid x^{(q)} = 0 \} \equiv \{ x \in H^q(C) \mid x = \text{cost.} \}$$

Their dimensions are equal to (2n) and to one, respectively, so that $N(D_q) \cap N(A) = \phi$ provided $n > 1$.

Hence, [2], the minimum problem:

$$\oint_C [\sigma^{(q)}(\tau)]^2 d\tau = \min_{x \in I_z} \oint_C [x^{(q)}(\tau)]^2 d\tau$$

$$I_z = \{ x \in H^q(C) \mid \langle K_i, x \rangle_{H^q} = x(\tau_i) = t_i ;\ \langle K_i', x \rangle_{H^q} = \qquad (3.3)$$

$$= x'(\tau_i) = \tau_i' ;\ 1 \le i \le n \}$$

$$z = (t_1, \ldots, t_n, t_1', \ldots, t_n') \in R^{2n}$$

has a unique solution $\sigma \in I_z \subset H^q(C)$, for any $z$, as long as $n > 1$.

This unique element of $I_z$ is herein called the Hermite interpolating closed spline corresponding to $[H^q(C), A, z]$.

The (2 n)- dimensional subspace $S_{HC} \subset H^q(C)$ of the Hermite closed splines is defined by:

$$S_{HC} = \{ s \in H^q(C) \mid \langle D_q s, D_q x \rangle_{H^0} = 0 ;\ \forall x \in N(A) = I_0 \} \qquad (3.4)$$

and, for any $z \in Z$, there exist a unique element $\sigma \in S_{HC}$ such that $A\sigma = z$. It has the extremal properties described by equations (2.8) and (2.9) [2].

Existence and uniqueness of the solution of the minimum problem (3.3) [and, hence; of the problem (2.1) ] is thus proved.

Proofs of the characterization and properties of the Hermite closed interpolating splines discussed in the previous paragraph hinge on the following theorem [2]:

Theorem 1

" $\sigma \in I_z$ is the closed Hermite interpolating function corresponding to ( $D_q, K_i, K_i', \tau_i, \tau_i'$ ) if there exist (2n) coefficients $\bar{\lambda}_i$ and $\bar{\lambda}_i'$ such that:

$$D_q' D_q \sigma = \sum_{i=1}^{n} \left( \bar{\lambda}_i K_i + \bar{\lambda}_i' K_i' \right) \in \left[ N(D_q) \right]^{\perp} \tag{3.5}$$

Here $D_q'$ denotes the adjoint of $D_q$ and $[\ \ ]^{\perp}$ the orthogonal subspace."

The demonstration, with few variations, is substantially the same as that employed in [1] and only the essential points will be detailed.

The definition of adjoint, eq. (3.5) with

$$\bar{\lambda}_i = (-1)^q \lambda_i \quad ; \quad \lambda_i' = (-1)^{q-1} \lambda_i'$$

and equation (3.1) lead to:

$$\langle D_q \sigma, D_q x \rangle_{H^0} = \sum_{i=1}^{n} (-1)^q \left[ \lambda_i x(\tau_i) - \lambda_i' x'(\tau_i) \right]$$

$$\forall x \in H^q(C)$$

from which one deduces that [1]:

a) the conditions imposed by theorem 1 reduce to the only requirement that $\sum_{i=1}^{n} (\bar{\lambda}_i K_i + \bar{\lambda}_i' K_i')$ be $H^q$ orthogonal to unity and this leas to:

$$\sum_{i=1}^{n} \lambda_i = 0 \tag{3.6}$$

b) Characterizing $\sigma^{(q)}$ amounts to finding a function $\psi \in H^{0}(C)$ which verifies the identity

$$\sum_{i=1}^{n} (-1)^{q} \left[ \lambda_i\, x(\tau_i) - \lambda_i'\, x'(\tau_i) \right] = \langle \psi, D_q x \rangle_{H^0}$$
$$\forall\, x \in H^{q}(C)$$

subject to the conditions (3.6) for then one has:

$$D_q \sigma = \sigma^{(q)} = \psi + \gamma_0 \tag{3.7}$$

with $\gamma_0$ an arbitrary constant and such an expression for $\sigma^{(q)}$ will contain (2n) arbitrary independent parameters.

Mac-Laurin developments of $x(\tau_i)$ and $x'(\tau_i)$ with the rest expressed in integral form yields:

$$x(\tau_i) = \sum_{j=0}^{q-1} \frac{\tau_i^{j}\, x^{(j)}(0)}{j!} + \oint_C \frac{(\tau_i - \tau)_+^{q-1}\, x^{(q)}(\tau)\, d\tau}{(q-1)!}$$

$$x'(\tau_i) = \sum_{j=1}^{q-1} \frac{\tau_i^{j-1}\, x^{(j)}(0)}{(j-1)!} + \oint_C \frac{(\tau_i - \tau)_+^{q-2}\, x^{(q)}(\tau)\, d\tau}{(q-2)!}$$

Account for the identity

$$(\tau - \tau_i)_+^{q-1} = (-1)^{q} (\tau - \tau_i)_+^{q-1} + (\tau_i - \tau)^{q-1}$$

apply the binomial formula to $(\tau_i - \tau)^{q-1}$ and $(\tau_i - \tau)^{q-2}$ and define:

$$\bar{\lambda}_m = \sum_{i=1}^{n} \frac{\bar{\lambda}_i\, \tau_i^{m}}{m!} \;;\quad \bar{\lambda}_m' = \sum_{i=1}^{n} \frac{\bar{\lambda}_i'\, \tau_i^{m-1}}{(m-1)!}$$

$$m \geq 0 \tag{3.8}$$

to obtain: [since $\bar{\alpha}_0 = 0$ upon eq. (3.6) and $\bar{\alpha}_0' = 0$ by definition]:

$$\langle \psi, D_q x \rangle_{H^0} = \sum_{i=1}^{n} \left[ \bar{\lambda}_i \, x(\tau_i) + \bar{\lambda}_i' \, x'(\tau_i) \right] =$$

$$= \sum_{j=1}^{q-1} \left[ \bar{\alpha}_j + \bar{\alpha}_j' \right] x^{(j)}(0) + \tag{3.9}$$

$$+ \oint_C \left\{ \sum_{i=1}^{n} \left[ \frac{\lambda_i (\tau - \tau_i)_+^{q-1}}{(q-1)!} + \frac{\lambda_i' (\tau - \tau_i)_+^{q-2}}{(q-2)!} + g(\tau) + f(\tau) \right] \right\} x^{(q)}(\tau)\, d\tau$$

with:

$$g(\tau) = \sum_{i=1}^{n} \frac{\bar{\lambda}_i (\tau_i - \tau)^{q-1}}{(q-1)!} = \sum_{j=0}^{q-2} (-1)^j \bar{\alpha}_{q-j-1} \frac{\tau^j}{j!}$$

$$f(\tau) = \sum_{i=1}^{n} \frac{\bar{\lambda}_i' (\tau_i - \tau)^{q-2}}{(q-2)!} = \sum_{j=0}^{q-2} (-1)^j \bar{\alpha}_{q-j-1}' \frac{\tau^j}{j!} \tag{3.10}$$

Only the first term on the right hand side of equation (3.9) must be further transformed to put it in the form of a scalar product in $H^0(C)$. This is readily accomplished since, as proven in [1], the (absolute) continuity of $x^k(\tau)$ for $1 \le k \le q-1$ implies that:

$$\sum_{j=1}^{q-1} (\bar{\alpha}_j + \bar{\alpha}_j') x^{(j)}(0) = \langle v, D_q x \rangle_{H^0} \; ; \quad \forall x \in H^0(C) \tag{3.11}$$

where:

$$v(\tau) = \sum_{j=1}^{q-1} \gamma_j \frac{\tau^j}{j!} \tag{3.12}$$

with the coefficient $\gamma_j$ such that:

$$\sum_{j=q-r}^{q-1} \frac{\gamma_j}{(r-q+j-1)!} = (-1)^{q+r+1}(\bar{\alpha}_r + \bar{\alpha}'_r) \tag{3.13}$$

$$1 \leq r \leq q-1$$

Substitution of eq. (3.11) into eq. (3.9) shows that:

$$\psi = \sum_{i=1}^{n} \left[ \frac{\lambda_i (\tau-\tau_i)_+^{q-1}}{(q-1)!} + \frac{\lambda'_i (\tau-\tau_i)_+^{q-2}}{(q-2)!} \right] + g(\tau) + f(\tau) + v(\tau)$$

Hence, on account of eq. (3.7), the following characterization of is obtained:

$$\sigma^{(q)} = \sum_{i=1}^{n} \left[ \frac{\lambda_i (\tau-\tau_i)_+^{q-1}}{(q-1)!} + \frac{\lambda'_i (\tau-\tau_i)^{q-2}}{(q-2)!} \right] + \sum_{j=0}^{q-1} \beta_{q+j} \frac{\tau^j}{j!} \tag{3.14}$$

where, upon eqs. (3.10) and (3.12):

$$\beta_{q+j} = \gamma_j + (-1)^j \left( \bar{\alpha}_{q-j-1} + \bar{\alpha}'_{q-j-1} \right) \tag{3.15}$$

$$0 \leq j \leq q-1$$

Of the (2n + q) coefficients ( $\beta_{q+j}, \lambda_i, \lambda'_i$ ) appearing in the expression (3.14) for $\sigma^{(q)}(\tau)$ only (2n) are independent. They are indeed related by eq. (3.6) and by the last (q-1) equations (3.15) which, on substituting $\gamma_j$ (for $j \neq 0$ ) from eq. (3.13) can be rewritten as:

$$\sum_{j=0}^{\tau-1} \left[ \frac{\beta_{2q-\tau+j}}{(j+1)!} + (-1)^{q+\tau+j} \frac{\bar{\alpha}_{\tau-j} + \bar{\alpha}'_{\tau-j}}{j!} \right] = 0 \tag{3.16}$$

$$1 \leq \tau \leq q-1$$

These equations for ( $2 \leq \tau \leq q-1$ ) express the vanishing of the discontinuities of $\sigma^{(q)}$ and of its first (q-3) derivatives at the closure point, as the following Lemma shows:

Lemma

" The function $\sigma^{(q)}(\tau)$ and its first ( q - 3 ) derivaties are continuous on C "

Proof

Given the properties of polynomials and of the functions $(\tau - \tau_i)_+^k$ the statement of the lemma needs to be proved only at the closure point $P_1$ . In the interval $]\, \tau_1 = 0, \tau_2 \,[$ eq. (3.14) reduces to

$$\sigma^{(q)}(\tau) = \sum_{j=0}^{q-1} \beta_{q+j} \frac{\tau^j}{j!} + \frac{\lambda_1 \tau^{q-1}}{(q-1)!} + \frac{\lambda'_1 \tau^{q-2}}{(q-2)!}$$

In the interval $]\tau_n, 1[$ it is $(\tau-\tau_i)_+ = (\tau-\tau_i)$ for any ( i ) so that, on account of eqs. (3.10), eq. (3.14) yields:

$$\sigma^{(q)}(\tau) = \sum_{j=0}^{q-1} \frac{\beta_{q+j}\,\tau^j}{j!} - \sum_{j=0}^{q-2} (-1)^j \left(\bar{\alpha}_{q-j-1} + \bar{\alpha}'_{q-j-1}\right) \frac{\tau^j}{j!}$$

The discontinuities of the (q + k)-th derivative at the closure point are therefore given by

$$\delta\sigma^{(q+k)}(P_1) = \sigma^{(q+k)}(0^+) - \sigma^{(q+k)}(1^-) =$$

$$= -\sum_{j=0}^{q-k-2} \left[ \frac{\beta_{q+k+j+1}}{(j+1)!} + (-1)^{j+k+1} \frac{\left(\bar{\alpha}_{q-k-j-1} + \bar{\alpha}'_{q-k-j-1}\right)}{j!} \right]$$

$$0 \le k \le q-3$$

or, with k + 1 = q - r:

$$\delta\sigma^{(2q-r-1)}(P_1) = -\sum_{j=0}^{r-1} \left[ \frac{\beta_{2q-r+j}}{(j+1)!} + (-1)^{q+r+j} \frac{\left(\bar{\alpha}_{r-j} + \bar{\alpha}'_{r-j}\right)}{j!} \right]$$

$$2 \le r \le q-1 \qquad (3.17)$$

and comparison with eq. (3.16) completes the proof of the lemma. The Hermite closed spline functions $s(\tau)$ corresponding to the set $\{\tau_i\}$ are obtained by integrating equation (3.14) q - times and using the (q) arbitrary constants to endorse the continuity of s ($\tau$) and of its first (q - 1) derivatives at the closure point so as to make $s \in H^q(C)$.

The result is the characterization of s ($\tau$) given by eq. (2.2). On proceeding as before one readily checks that the discontinuity of the

(2q - r - 1)-the derivatives at the closure point $P_1$ are still given by eqs. (3.17) with $r$ now ranging from 2 to (2q - 1). Hence imposing the continuity of s $(\tau)$ and of its first (q - 1) derivatives amounts to the statement that eqs. (3.16) must hold also for (r) ranging from the value (q) [corresponding to the condition $\delta s^{(q-1)}(P_1)=0$] to the value (2q - 1) [corresponding to the condition $\delta s(P_1)=0$]. Equations (2.4) are thus proved.

This completes the proof of the statements made in paragraph (2).

In particular, equation (2.3) is the condition (3.6) and equation (2.7) is the definition (3.4) of the subspace $S_{HC}$ .

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The report gives the appropriate reference of papers originated under the subject Grant. In addition, it presents a concise summary of motivation and results of work already available in published form (Part I) and, in extenso, the theory and properties of a new class of closed spline functions (Part II): the Hermite Closed Splines.